JOURNAL OF FUNCTIONAL ANALYSIS **141**, 510–539 (1996)
ARTICLE NO. 0140

Ultracontractivity and Nash Type Inequalities*

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Received July 22, 1995

FOR ALBERT BADRIKIAN

I. INTRODUCTION

that

$$\|T_t\|_{1\to\infty} \leqslant m(t), \qquad \forall t>0. \tag{$*$}$$

In the context of heat kernel estimates on manifolds and Lie groups, it has been very useful to translate estimates of this type in terms of coercivity inequalities involving the infinitesimal generator of T_t (see [D] and [VSC]). On unimodular and amenable non-compact Lie groups, the only functions that appear are of the type $m(t)=Ct^{-d/2}$, $d\in\mathbb{N}^*$, for small t, and $m(t)=Ct^{-D/2}$, $D\in\mathbb{N}^*$, or $m(t)=Ce^{-ct^{1/3}}$, for large t (see [VSC]). But on non-compact Riemannian manifolds, the large time behaviour of the heat kernel reflects the geometry at infinity of the manifold, and cannot be controlled universally by a limited class of functions.

For the purpose of this introduction, suppose T_t symmetric Markovian, and let $-A$ be its generator. In the case where $m(t)$ is of the form $Ct^{-n/2}$, $n>2$, $(*)$ was proved in [V] to be equivalent to the Sobolev inequality

$$\|f\|^2_{2n/(n-2)} \leqslant C'(Af, f), \qquad \forall f\in\mathscr{D}(A);$$

see also [C1] and [C2]. In [CKS], $(*)$ for $m(t)=Ct^{-n/2}$, $n>0$, was proved to be equivalent to the apparently weaker Nash inequality

$$\|f\|_2^{2+(4/n)} \leqslant C'\ \|f\|_1^{4/n}\,(Af, f), \qquad \forall f\in\mathscr{D}(A).$$

* This paper was partially written during a stay at the Australian National University, Canberra.

0022-1236/96 $18.00

Note that the equivalence between the Sobolev and Nash inequalities for $n > 2$ has recently been proved directly in [BCLS].

The methods of [CKS] were pushed further in [T], and those of [C1] in [CM], and more general functions m could be treated (see also [Carr]), however always under the assumption $m(t) \geqslant ct^{-\eta}$, for some positive η.

By using families of logarithmic Sobolev inequalities with parameter, [D] reaches a much larger class of functions m, including for example the functions $e^{-ct^{\alpha}}$, $0 < \alpha < 1$; for functions m whose logarithm at t is comparable to its mean from 0 to t (but other examples can be treated), $(*)$ is shown to be equivalent to

$$\int f^2 \log \frac{f^2}{\|f\|_2^2} \leqslant \varepsilon(Af, f) + \log m(\varepsilon) \, \|f\|_2^2, \qquad \forall f \in \mathscr{D}(A)$$

(modulo identification of m and $Cm(c.)$). Note that one can also use one-parameter families of regular Sobolev inequalities (see the Appendix below).

Our aim here is to show that the estimate $\|T_t\|_{1 \to \infty} \leqslant m(t)$ is also equivalent to a Nash type inequality of the form

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A), \ \|f\|_1 = 1,$$

for every function m whose logarithmic derivative has polynomial growth. This is nothing but the abstract version of a theorem due to Grigor'yan [G1] that treats the case where T_t is the heat semigroup on a Riemannian manifold. Our proof is close in spirit to the one of Grigor'yan; however, a crucial part of the latter relies upon constructions that are specific to the setting of second order differential operators, like the Neumann heat semigroup associated with an open set and the Sturm-Liouville expansion of its kernel. So, as it happens, to extend this proof we had to simplify it. By the same token, we get a discrete time version. All this can be seen as a wide generalisation of [V] and [CKS].

Compared to the method using logarithmic Sobolev inequalities, this method is probably easier to handle in the discrete time setting, and more importantly it has a clear interpretation in isoperimetric terms (see [G1], [C6] and Section IV below). This enables us for example to give geometric conditions that control the decay of Markov chains on graphs, which generalise what was known in the case of polynomial decay [V0]. We also get several applications to the large time decay of the heat kernel on Riemannian manifolds, among which the invariance under rough isometry. The class of functions m we treat is certainly sufficient for the applications, but does not quite cover the class treated in [D].

II. ULTRACONTRACTIVITY AND NASH TYPE INEQUALITIES

From now on, (X, ξ) will be a σ-finite measure space and L^p will mean $L^p(X, \xi)$.

The following proposition is not original. It is explicitly contained in [T], and anyway directly stems from the ideas of Nash [N], as elaborated in [CKS]; see also [G1]. We give the proof for the sake of completeness.

II.1. PROPOSITION. *Let T_t be a semigroup on L^p, $1 \leqslant p \leqslant +\infty$, with infinitesimal generator $-A$. Suppose that T_t is equicontinuous on L^1 and L^∞, i.e.,*

$$\sup_t \|T_t\|_{1 \to 1}, \sup_t \|T_t\|_{\infty \to \infty} \leqslant M < +\infty,$$

and that

$$\theta(\|f\|_2^2) \leqslant \operatorname{Re}(Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant M,$$

where θ: $]0, +\infty[\to]0, +\infty[$ is continuous and satisfies $\int^{+\infty} dx/\theta(x) < +\infty$. Then T_t is ultracontractive and

$$\|T_t\|_{1 \to \infty} \leqslant m(t), \qquad \forall t > 0,$$

where m is the solution of

$$-m'(t) = \theta(m(t))$$

on $]0, +\infty[$ such that $m(0) = +\infty$, or alternatively the inverse function of $p(t) = \int_t^{+\infty} dx/\theta(x)$.

Proof. Since $\|T_t f\|_1 \leqslant M\|f\|_1$, replacing f by $T_t f$ in the hypothesis gives

$$\theta(\|T_t f\|_2^2) \leqslant -\frac{1}{2}\frac{d}{dt}\|T_t f\|_2^2, \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 = 1.$$

Fix $f \in \mathscr{D}(A)$ such that $\|f\|_1 = 1$, and let $I(t) = \|T_t f\|_2^2$. One has

$$I'(t) \leqslant -2\theta(I(t)), \qquad t \geqslant 0,$$

which gives by integration

$$\int_{I(t)}^{I(0)} \frac{dx}{\theta(x)} \geqslant 2t.$$

Now one has by definition

$$\int_{m(2t)}^{+\infty} \frac{dx}{\theta(x)} = 2t.$$

This yields $I(t) \leqslant m(2t)$, and

$$\|T_t\|_{1\to 2}^2 \leqslant m(2t).$$

The same argument works for T_t^* as well, so that

$$\|T_t\|_{2\to\infty}^2 \leqslant m(2t),$$

and $\|T_t\|_{1\to\infty} \leqslant \|T_{t/2}\|_{1\to 2} \|T_{1/2}\|_{2\to\infty}$, whence the conclusion.

Remark. If T_t is symmetric Markov, one has $\|T_t f\|_1 = \|T_t^* f\|_1 = \|f\|_1$, $\forall f \in L^1_+$. Hence in that case the slightly weaker inequality

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ f \geqslant 0,\ \|f\|_1 = 1$$

is sufficient.

The main point of this paper is to give an exact converse to Proposition II.1 for symmetric semigroups; for example, given θ, [T] recovers the initial m only if $t\, m(t)$ is bounded below (in fact, one can probably make this proof work then $t^\alpha m(t)$ is bounded below for some $\alpha > 0$, which is also the scope of [CM]). In fact, the natural thing to do, starting from

$$\|T_t\|_{1\to\infty} \leqslant m(t), \qquad \forall t > 0,$$

is to write

$$\begin{aligned} \|f\|_2 &\leqslant \|f - T_t f\|_2 + \|T_t f\|_2 \\ &\leqslant c\sqrt{t}\,(Af, f)^{1/2} + \sqrt{m(t)}\,\|f\|_1, \end{aligned}$$

and to choose $m(t) = ((\|f\|_2)/(2\,\|f\|_1))^2$. This gives

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 = 1,$$

where $\theta(x)$ is comparable to $x/(m^{-1}(x))$, i.e., $\theta(m(x))$ comparable to $(m(x))/x$ (this is more or less the argument of [CKS]). But, to close up the circle with Proposition II.1, we need instead $\theta(m(x))$ to be comparable to $-m'(x)$, which is not the same unless m is a negative power.

This will be achieved in two steps: first, using more specifically than in the above argument the fact that T_t is a semigroup, we shall prove a Nash type inequality with $\theta(x) = \sup_{t>0} (x/t) \log(x/(m(t)))$, without any assumption on m. Then, following [G1], we shall exhibit a regularity condition on the decay of m that allows one to take θ comparable to $-m' \circ m^{-1}$.

II.2. PROPOSITION. *Let T_t be a symmetric contractive semigroup on L^2, with infinitesimal generator $-A$, that satisfies*

$$\|T_t\|_{1 \to 2}^2 \leqslant m(t), \qquad \forall t > 0.$$

Then

$$\tilde{\theta}(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1,$$

where $\tilde{\theta}(x) = \sup_{t>0} (x/2t) \log(x/(m(t)))$.

Proof. Let $\int_0^{+\infty} \lambda\, dE_\lambda$ be the spectral resolution of A. Then $T_t = \int_0^{+\infty} e^{-\lambda t}\, dE_\lambda$. Let $f \in \mathscr{D}(A) \backslash \{0\}$. Since

$$\int_0^{+\infty} \frac{(dE_\lambda f, f)}{\|f\|_2^2} = 1,$$

Jensen yields

$$\exp\left(-2 \int_0^{+\infty} \lambda \frac{(dE_\lambda f, f)}{\|f\|_2^2}\, t\right) \leqslant \int_0^{+\infty} e^{-2\lambda t} \frac{(dE_\lambda f, f)}{\|f\|_2^2},$$

in other terms

$$\exp\left(-2 \frac{(Af, f)}{\|f\|_2^2}\, t\right) \leqslant \frac{\|T_t f\|_2^2}{\|f\|_2^2}.$$

Now, by hypothesis, $\|T_t f\|_2^2 \leqslant m(t)\, \|f\|_1^2$. Fix $f \in \mathscr{D}(A) \backslash \{0\}$ such that $\|f\|_1 \leqslant 1$. One has

$$\exp\left(-2 \frac{(Af, f)}{\|f\|_2^2}\, t\right) \leqslant \frac{m(t)}{\|f\|_2^2}, \qquad \forall t > 0,$$

hence

$$\frac{(Af, f)}{\|f\|_2^2} \geqslant \frac{1}{2t} \log \frac{\|f\|_2^2}{m(t)}, \qquad \forall t > 0.$$

This proves the Proposition.

Remarks. — The crucial inequality here is

$$\exp(-(Af, f)\, t) \leqslant \|T_t f\|_2,\ \|f\|_2 = 1.$$

One can also see it as a consequence of the log-convexity of $t \to \|T_t f\|_2^2$. To try and weaken the symmetry assumption, one may wonder for which semigroups one has

$$\exp(-\mathrm{Re}(Af, f)\, t) \leqslant C\, \|T_t f\|_2, \; \|f\|_2 = 1.$$

It would be sufficient to have

$$\frac{\mathrm{Re}(AT_t f, T_t f)}{\|T_t f\|_2^2} \leqslant C \frac{\mathrm{Re}(Af, f)}{\|f\|_2^2}, \qquad t > 0.$$

— It follows from the proof that $\tilde{\theta}(x)$ is always finite, therefore the hypothesis cannot hold unless $(\log m(t))/t$ is bounded below.

Let us say that a differentiable function m defined on a subinterval of $\mathbb{R}_+$ and with positive values satisfies condition (D) if its logarithmic derivative has polynomial growth, i.e. $M(t) = -\log m(t)$ is such that

$$M'(u) \geqslant \alpha M'(t), \qquad \forall t > 0, \quad \forall u \in [t, 2t],$$

for some $\alpha > 0$. For instance, functions $m(t)$ that behave like $t^{-n/2}$, $n > 0$, for small t, and e^{-ct^α}, $0 \leqslant \alpha \leqslant 1$, for large t satisfy (D); notice that they also satisfy (D) for $\alpha > 1$, but we have just noticed that $m(t)$ cannot be so small in the problem under consideration.

II.3. Lemma ([G1]). *Let m be a decreasing differentiable bijection of $\mathbb{R}_+^*$ satisfying (D). Then $\tilde{\theta}(x) = \sup_{t>0} (x/2t) \log(x/(m(t))) \geqslant\; = -cm'(m^{-1}(x))$, with $c > 0$.*

Proof. It suffices to show that for $u > 0$,

$$\sup_{t>0} \frac{m(u)}{t} \log \frac{m(u)}{m(t)} \geqslant -cm'(u).$$

Choose $t = 2u$. Then

$$\frac{1}{t} \log \frac{m(u)}{m(t)} = \frac{1}{2u} (\log m(u) - \log m(2u)).$$

Let $v \in \,]u, 2u[\,$ be such that

$$\frac{1}{u} (\log m(u) - \log m(2u)) = \frac{-m'(v)}{m(v)}.$$

Since m satisfies (D),

$$\frac{-m'(v)}{m(v)} \geqslant \alpha \frac{-m'(u)}{m(u)}, \qquad \forall u > 0.$$

Therefore

$$\frac{m(u)}{t} \log \frac{m(u)}{m(t)} \geqslant -\frac{\alpha}{2} m'(u).$$

We have proved the Lemma.

II.4. PROPOSITION. *Let T_t be a symmetric contractive semigroup on L^2, with infinitesimal generator $-A$, such that*

$$\|T_t\|_{1\to 2}^2 \leqslant m(t), \qquad \forall t > 0,$$

where m is a decreasing differentiable bijection of $\mathbb{R}_+^*$ satisfying (D). Then

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1,$$

where $\theta(x) = -cm'(m^{-1}(x))$.

Note that under the assumptions of Proposition II.4, $\int^{+\infty} dx/\theta(x) < +\infty$, since $m(t)$ goes to $+\infty$ as t goes to zero.

Let m_1, m_2 be two functions from $]0, +\infty[$ to itself; we shall say that $m_1 \preccurlyeq m_2$ if there exists $C, C' > 0$ such that $m_1(t) \leqslant C m_2(C't)$, and that m_1, m_2 are equivalent ($m_1 \simeq m_2$) if $m_1 \preccurlyeq m_2$ and $m_2 \preccurlyeq m_1$. Note that if m_i, $i = 1, 2$ are decreasing differentiable bijections and if $\theta_i(x) = -m_i'(m_i^{-1}(x))$, then $m_1 \simeq m_2$ if and only if $\theta_1 \simeq \theta_2$. In the two following statements, the inequalities will be written modulo equivalence of functions. Note that to have a neat theory we will suppose m differentiable or C^1, even in Section IV where we consider discrete times! Of course these assumptions should not be taken too seriously: one can always regularise m, and get an equivalent function.

Proposition II.1, the remark that follows, and Proposition II.4 yield

II.5. THEOREM. *Let T_t be a symmetric Markov semigroup, with infinitesimal generator $-A$, m a decreasing C^1 bijection of $\mathbb{R}_+^*$ satisfying (D), and $\theta(x) = -m'(m^{-1}(x))$. Then the following conditions are equivalent*:

(i) *T_t is ultracontractive and $\|T_t\|_{1\to\infty} \leqslant m(t)$, $\forall t > 0$.*

(ii) *$\theta(\|f\|_2^2) \leqslant (Af, f)$, $\forall f \in \mathscr{D}(A)$, $\|f\|_1 = 1$.*

Application. It follows from [BLCS], cor. 7.3 and prop. 10.3, that (ii) is equivalent to the Faber-Krahn type inequality

$$\|f\|_2 \leqslant \psi(|\Omega|)(Af, f)^{1/2}, \qquad \forall f \in \mathscr{D}(A),$$

where Ω contains the support of f, $|\Omega| = \xi(\Omega)$ is its measure, and $\theta(x) = x/(\psi^2(1/x))$ or $\psi(x) = 1/(\sqrt{x\theta(1/x)})$. When, for instance, $A = -\Delta$ is the Laplace-Beltrami operator on a Riemannian manifold M, ξ the Riemannian measure, and $p_t(x, y) = e^{-t\Delta}(x, y)$ the heat kernel, one concludes from II.4 that, if m satisfies (D),

$$\sup_{x,\, y \in M} p_t(x, y) \leqslant m(t), \qquad \forall t > 0, \tag{1}$$

if and only if

$$\|f\|_2 \leqslant \psi(|\Omega|)\, \|\, |\nabla f|\, \|_2, \qquad \forall f \in C_0^\infty(M), \tag{2}$$

where Ω ranges over compact domains of M with smooth boundary, $|\Omega| = \xi(\Omega)$, f is supported in Ω, and

$$\psi(x) = \frac{1}{\sqrt{-xm'(m^{-1}(1/x))}}$$

or

$$m^{-1}(t) = \int_t^{+\infty} \psi^2\left(\frac{1}{x}\right) \frac{dx}{x}.$$

The inequality (2) can be reformulated as

$$\frac{1}{\psi^2(|\Omega|)} \leqslant \lambda_1(\Omega), \tag{2$'$}$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the domain Ω.

Note that (2), therefore (1), is implied by the isoperimetric inequality

$$\frac{|\Omega|}{\psi(|\Omega|)} \leqslant |\partial\Omega|, \tag{3}$$

where Ω ranges over compact domains of M with smooth boundary. Conversely, if ψ is such that $\psi(2x) \leqslant C\psi(x)$, (1) implies the volume lower bound

$$\xi(B(x, r)) \geqslant \psi^{-1}(r), \qquad \forall r > 0,$$

but the connection is not clear in general. For all this, see [C6]. The equivalence between (1) and (2)′ was first proved in [G1].

One can also use Proposition II.1 and Proposition II.2 for two different semigroups and formulate

II.5. THEOREM. *Let T_t be a symmetric Markov semigroup, with infinitesimal generator $-A$ and let S_t be a semigroup that is equicontinuous on L^1 and L^∞, with infinitesimal generator $-B$. Suppose that $\mathcal{D} = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ is dense in L^2 and that*

$$(Af, f) \leqslant C \operatorname{Re}(Bf, f), \qquad \forall f \in \mathcal{D}.$$

If T_t is ultracontractive with

$$\|T_t\|_{1 \to \infty} \leqslant m(t),$$

and if m is a decreasing C^1 bijection of $\mathbb{R}_+^*$ satisfying (D), then S_t is also ultracontractive and

$$\|S_t\|_{1 \to \infty} \leqslant m(t).$$

Remarks. — Corollary II.5 also follows from [D], Thm. 2.2.3 and Cor. 2.2.8, at least when S_t is symmetric. The assumption on m is then

$$\frac{1}{t} \int_0^t \log m(s)\, ds \geqslant c \log m(t).$$

However, examples of functions m not satisfying this assumption are also treated there.

— In the case where $m(t) = e^{-\lambda t} t^{-n/2}$, with λ the spectral gap of A, the natural question is whether

$$\|S_t\|_{1 \to \infty} \leqslant C' e^{-\mu t} t^{-n/2},$$

with μ the spectral gap of B. Though m satisfies (D), the above result does not answer this question. In fact, the comparability of Dirichlet forms cannot ensure the stability of such a behaviour of the associated semigroups, as is noticed in [L] (for an interesting example in the setting of finitely generated groups, and with discrete time, see [Cart]). However, stability does hold at the expense of an equivalence between expressions involving higher powers of the generators (see [C4]).

III. LOCALISATION AT INFINITY

Here localisation at infinity means restriction to large time: we want to characterise the behaviour $\|T_t\|_{1\to\infty} \leqslant m(t)$, $\forall t \geqslant t_0$, $t_0 > 0$. This is useful in the study of the heat kernel on Riemannian manifolds (see [C1], [C2], and [G2]). The modification we impose on our Nash inequalities is slightly different from the one that appears in [CKS] for the polynomial case; we simply restrict them to functions such that $\|f\|_2/\|f\|_1$ is bounded above. This means that their local singularities are controlled: in some sense, the localisation in time is also a localisation in space. For simplicity we stick to the symmetric Markovian case.

III.1. PROPOSITION. *Let T_t be a symmetric Markov semigroup, with infinitesimal generator $-A$, such that*

$$\|T_t\|_{1\to\infty} \leqslant m(t), \qquad \forall t \geqslant t_0,$$

where $t_0 > 0$ and $m\colon [t_0, +\infty[\to]0, m(t_0)]$ is a decreasing differentiable bijection satisfying (D). Then

$$\theta(\|f\|_2^2) \leqslant C(Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1,\ \|f\|_2 \leqslant a,$$

where $\theta(x) = -m'(m^{-1}(x))$ and $a^2 = m(t_0)$.

Proof. The same arguments as in II.2 show that

$$\frac{(Af, f)}{\|f\|_2^2} \geqslant \frac{1}{2t} \log \frac{\|f\|_2^2}{m(t)}, \qquad \forall t \geqslant t_0.$$

Now, if $x \leqslant m(t_0)$, one sees as in II.3 that

$$\sup_{t \geqslant t_0} \frac{x}{t} \log \frac{x}{m(t)} \geqslant -cm'(m^{-1}(x)).$$

III.2. PROPOSITION. *Let T_t be a symmetric Markov semigroup, with infinitesimal generator $-A$, such that $\|T_{t_0}\|_{1\to 2} = a < +\infty$. Suppose that*

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1,\ \|f\|_2 \leqslant a,$$

where $a > 0$ and $\theta\colon]0, a^2] \to]0, +\infty[$ is continuous. Then

$$\|T_t\|_{1\to\infty} \leqslant m(t - 2t_0), \qquad \forall t > 2t_0,$$

where m is the solution of

$$-m'(t) = \theta(m(t))$$

on $[0, +\infty[$ *such that* $m(0)=a^2$, *or alternatively the inverse function of* $p(t)=\int_t^{a^2} dx/\theta(x)$.

Proof. Let $f\in\mathscr{D}(A)$ be such that $\|f\|_1\leqslant 1$, and $\|f\|_2\leqslant a$. Since T_t is symmetric Markov, one has $\|T_t f\|_1\leqslant 1$ and $\|T_t f\|_2\leqslant a$ for $t>0$ as well, thus

$$\theta(\|T_t f\|_2^2)\leqslant (AT_t f, T_t f), \qquad \forall t>0.$$

Proceeding as in II.1, one sees that for such f, $\|T_t f\|_2^2\leqslant m(2t)$. Now, for any $f\in\mathscr{D}(A)$ such that $\|f\|_1\leqslant 1$, one has $\|T_{t0} f\|_1\leqslant 1$ and $\|T_{t0} f\|_2\leqslant a$. Therefore $\|T_{t+t0} f\|_2^2\leqslant m(2t)$. This gives $\|T_{t+t0}\|_{1\to 2}=\|T_{t+t0}\|_{2\to\infty}\leqslant\sqrt{m(2t)}$, hence $\|T_{t+2t0}\|_{1\to\infty}\leqslant m(t)$. This proves the proposition.

The following statement is formulated modulo equivalence of functions in the sense of Section II.

III.3. THEOREM. *Let* T_t *be a symmetric Markov semigroup, with infinitesimal generator* $-A$. *Suppose that* $\|T_{t0}\|_{1\to+\infty}=a^2<+\infty$. *Let* m *be a decreasing* C^1 *bijection of* $\mathbb{R}_+^*$ *satisfying* (D), *and* $\theta(x)=-m'(m^{-1}(x))$. *Then the following conditions are equivalent*:

(i) $\|T_t\|_{1\to\infty}\leqslant m(t)$, $\forall t\geqslant t_0$.

(ii) $\theta(\|f\|_2^2)\leqslant (Af, f)$, $\forall f\in\mathscr{D}(A)$, $\|f\|_1\leqslant 1$, $\|f\|_2\leqslant a$.

A by-product of the above proposition is a proof of the following fact: if T_t is a symmetric Markov semigroup that is regularising in the sense that, for some $t_0>0$, $\|T_{t0}\|_{1\to\infty}<+\infty$, then $\|T_t\|_{1\to\infty}$ has a subexponential behaviour, i.e., $\|T_t\|_{1\to\infty}\leqslant e^{-ct}$, for all $t\geqslant 1$ and some $c>0$, if and only if A has an L^2 spectral gap, i.e., $\lambda\|f\|_2^2\leqslant (Af, f)$, $\forall f\in\mathscr{D}(A)$, for some positive λ.

Application. Returning to the case of the heat kernel on a Riemannian manifold M, it follows from this paragraph that, if M satisfies some weak local geometry assumptions (for example $(DV)_0$, $(P)_0$ and $\inf_{x\in M} V(x, 1)>0$, see Section VI below), and if m satisfies (D), then

$$\sup_{x,\, y\in M} p_t(x, y)\leqslant m(t), \qquad \forall t\geqslant 1 \tag{$1)_\infty$}$$

if and only if

$$\|f\|_2\leqslant\psi(|\Omega|)\,\|\,|\nabla f|\,\|_2, \qquad \forall f\in C_0^\infty(\Omega) \tag{$2)_\infty$}$$

or

$$\frac{1}{\psi^2(|\Omega|)} \leqslant \lambda_1(\Omega) \tag{2$'_\infty$}$$

where Ω ranges over compact domains of M with smooth boundary containing a geodesic ball of radius one. This in turn follows from what Chavel and Feldman [CF] call (in the polynomial case) a modified isoperimetric inequality

$$\frac{|\Omega|}{\psi(|\Omega|)} \leqslant |\partial\Omega|, \tag{3}$$

where Ω ranges over compact domains of M with smooth boundary containing a geodesic ball of radius one. One therefore recovers essentially Theorem 1.2 from [G2]. We leave the proof to the reader. Anyway, these results also follow from the discretisation techniques from Section VI.

IV. REGULARISING OPERATORS

Let T be a linear operator acting on the $L^p(X, \xi)$, $1 \leqslant p \leqslant +\infty$. We say that T is regularising if it is bounded from L^p to L^p, $1 \leqslant p \leqslant +\infty$ and from L^1 to L^∞, hence from L^p to L^q if $1 \leqslant p \leqslant q \leqslant +\infty$. For such operators, the connection between estimates of the form

$$\|T^k\|_{1 \to \infty} \leqslant Ck^{-n/2}$$

and Sobolev type inequalities has been studied in [V0], [CS1], [CS2]. The connection between such estimates and Nash type inequalities has been studied in [CKS], [CS1], [CS2].

IV.1. Proposition. *Let T be a regularising operator. Suppose that T is power-bounded on L^1, i.e.,*

$$\sup_{k \in \mathbb{N}^*} \|T^k\|_{1 \to 1} \leqslant M, \qquad 1 \leqslant M < +\infty,$$

and that

$$\theta(\|Tf\|_2^2) \leqslant \|f\|_2^2 - \|Tf\|_2^2, \qquad \forall f \in L^1 \cap L^2, \|f\|_1 \leqslant M,$$

where $a=\|T\|_{1\to 2}$, θ: $]0, M^2a^2] \to]0, +\infty[$ *is continuous, non-decreasing, and* $\lim_{x\to 0^+}\theta(x)=0$. *Then*

$$\|T^k\|_{1\to 2}^2 \leqslant m(k), \qquad \forall k\in\mathbb{N}^*,$$

where m is defined by $\theta(m(k+1))=m(k)-m(k+1)$ *and* $m(1)=a^2$.

Proof. Observe first that m is well-defined from $\mathbb{N}^*$ to $]0, a^2]$, since $x\to\theta(x)+x$ is strictly increasing on $]0, a^2]$ and its image contains $]0, a^2]$. Let f be such that $\|f\|_1=1$. Set $u_k=\|T^kf\|_2^2$; since $\|T^kf\|_1\leqslant M$, one has by hypothesis $\theta(u_{k+1})\leqslant u_k-u_{k+1}$, $\forall k\in\mathbb{N}^*$, and $u_1\leqslant m(1)$. Then $u_k\leqslant m(k)$ and $u_{k+1}>m(k+1)$ would imply

$$\theta(u_{k+1})\geqslant\theta(m(k+1))=m(k)-m(k+1)>u_k-u_{k+1},$$

a contradiction. Therefore $u_k\leqslant m(k)$, $\forall k\in\mathbb{N}^*$, which proves the proposition.

Remark. In the special case where T is symmetric Markov, one sees along the same lines that

$$\theta(\|Tf\|_2^2)\leqslant\|f\|_2^2-\|Tf\|_2^2, \qquad \forall f\in L^1\cap L^2,\ \|f\|_1=1,\ f\geqslant 0,$$

implies

$$\|T^{2k}\|_{1\to\infty}\leqslant m(k), \qquad \forall k\in\mathbb{N}^*.$$

IV.2. Proposition. *Let T be a regularising self-adjoint contraction on* L^2, *such that*

$$\|T^k\|_{1\to 2}^2 \leqslant m(k), \qquad \forall k\in\mathbb{N}^*,$$

where m is a differentiable decreasing bijection from $[0, +\infty[$ *to* $]0, m(0)]$ *satisfying* (D). *Then, if R is a regularising operator,*

$$\theta(c\,\|Rf\|_2^2)\leqslant C(\|f\|_2^2-\|TF\|_2^2), \qquad \forall f\in L^1\cap L^2,\ \|f\|_1\leqslant 1,$$

where $\theta(x)=-m'(m^{-1}(x))$.

Proof. Set $U=I-T^2$; U is a self-adjoint contraction of L^2 with non-negative spectrum. Let $\int_0^1\lambda\,dE_\lambda$ be its spectral resolution. For $k\in\mathbb{N}^*$, write

$$T^{2k}=(I-U)^k=\int_0^1(1-\lambda)^k\,dE_\lambda,$$

hence, for $f \in L^2$,

$$\| T^k f \|_2^2 = (T^{2k} f, f) = \int_0^1 (1-\lambda)^k \, (dE_\lambda f, f).$$

Since $\lambda \to (1-\lambda)^k$ is convex, Jensen yields, if $f \neq 0$,

$$\left(1 - \frac{\int_0^1 \lambda (dE_\lambda f, f)}{\| f \|_2^2}\right)^k \leqslant \int_0^1 (1-\lambda)^k \, \frac{(dE_\lambda f, f)}{\| f \|_2^2}.$$

This means that

$$\left(\frac{\| Tf \|_2^2}{\| f \|_2^2}\right)^k \leqslant \frac{\| T^k f \|_2^2}{\| f \|_2^2}.$$

Fix $f \in L^2 \backslash \{0\}$ such that $\| f \|_1 \leqslant 1$; by hypothesis, $\| T^k f \|_2^2 \leqslant m(k)$, therefore

$$\left(\frac{\| Tf \|_2^2}{\| f \|_2^2}\right)^k \leqslant \frac{m(k)}{\| f \|_2^2},$$

i.e., if $Tf \neq 0$,

$$\log \frac{\| f \|_2^2}{\| Tf \|_2^2} \geqslant \frac{1}{k} \log \frac{\| f \|_2^2}{m(k)}.$$

Now

$$\frac{\| f \|_2^2 - \| Tf \|_2^2}{\| Tf \|_2^2} \geqslant \log \frac{\| f \|_2^2}{\| Tf \|_2^2},$$

and therefore

$$\| f \|_2^2 - \| Tf \|_2^2 \geqslant \frac{\| Tf \|_2^2}{k} \log \frac{\| f \|_2^2}{m(k)}.$$

Now, if $\| Tf \|_2^2 \geqslant \frac{1}{2} \| f \|_2^2$,

$$\| f \|_2^2 - \| Tf \|_2^2 \geqslant \tfrac{1}{2} \, \tilde{\theta}(\| f \|_2^2),$$

where $\tilde{\theta}(x) = \sup_{k \in \mathbb{N}^*} (x/k) \log(x/(m(k)))$. Since $\tilde{\theta}$ is non-decreasing and R is bounded on L^2, it follows that

$$\| f \|_2^2 - \| Tf \|_2^2 \geqslant \tfrac{1}{2} \, \tilde{\theta}(c \, \| Rf \|_2^2),$$

where $c = \|R\|_{2\to 2}^{-2}$. If $\|f\|_2^2 - \|Tf\|_2^2 \geqslant 1$, since R is regularising one has

$$\tilde{\theta}(\|Rf\|_2^2) \leqslant \tilde{\theta}(\|R\|_{1\to 2}^2) \leqslant \tilde{\theta}(\|R\|_{1\to 2}^2)(\|f\|_2^2 - \|Tf\|_2^2).$$

Finally, if $\|Tf\|_2^2 \leqslant \frac{1}{2}\|f\|_2^2$ and $\|f\|_2^2 - \|Tf\|_2^2 \leqslant 1$, it follows that

$$\tfrac{1}{2}\|f\|_2^2 \leqslant \|f\|_2^2 - \|Tf\|_2^2 \leqslant 1,$$

and therefore

$$\tilde{\theta}(c\,\|Rf\|_2^2) \leqslant \tilde{\theta}(\|f\|_2^2) \leqslant C\,\|f\|_2^2 \leqslant 2c(\|f\|_2^2 - \|Tf\|_2^2).$$

Here we use the fact that, if $x \leqslant 2$,

$$\tilde{\theta}(x) = \sup_{k\in\mathbb{N}^*} \frac{x}{k}\log\frac{x}{m(k)} \leqslant \sup_{k\in\mathbb{N}^*} \frac{x}{k}\log\frac{2}{m(k)} = \frac{x}{2}\,\tilde{\theta}(2).$$

We have proved

$$\tilde{\theta}(c\,\|Rf\|_2^2) \leqslant C(\|f\|_2^2 - \|Tf\|_2^2), \qquad \forall f \in L^1 \cap L^2,\ \|f\|_1 \leqslant 1.$$

Note that one can always take c small enough so that $c\,\|Rf\|_2^2 \leqslant m(1)$ if $\|f\|_1 \leqslant 1$. Then, an easy adaptation of Lemma II.3 shows that $\tilde{\theta}$ is comparable to $-m' \circ m^{-1}$ on $]0, m(1)]$.

This proves the proposition.

Remark. It is clear that the only rôle of the symmetry assumption is to provide the inequality

$$(Tf, Tf) \leqslant \|T^2 f\|_2\,\|f\|_2,$$

from which the result follows: one can then prove by recurrence that

$$\left(\frac{\|Tf\|_2^2}{\|f\|_2^2}\right)^k \leqslant \frac{\|T^k f\|_2^2}{\|f\|_2^2}$$

(see [I], Section 7.1; the operators satisfying this inequality are called paranormal). I owe this remark to Derek Robinson.

Let us say that a one-to-one map m from $\mathbb{R}_+$ to itself satisfies condition $(\tilde{D})$ if it is log-convex, decreasing, differentiable, if it satisfies (D), and moreover $\inf_{k\in\mathbb{N}^*}((m'(k+1))/(m'(k))) > 0$. Note that the examples we gave in Section II also satisfy $(\tilde{D})$, for $0 < \alpha \leqslant 1$.

The following statement is formulated up to equivalence of functions in the sense of Section II.

IV.3. THEOREM. *Let T be a regularising self-adjoint Markov operator, let m be a C^1 function satisfying $(\tilde{D})$, and let $\theta(x) = -m'(m^{-1}(x))$. Then the following conditions are equivalent*:

(i) $\|T^k\|_{1\to\infty} \leqslant m(k)$, $\forall k \in \mathbb{N}^*$,

(ii) $\theta(\|Tf\|_2^2) \leqslant \|f\|_2^2 - \|Tf\|_2^2$, $\forall f \in L^1 \cap L^2$, $\|f\|_1 = 1$.

Proof. The implication from (i) to (ii) follows from IV.2. Suppose (ii). Under our assumptions on m, θ is continuous, non-decreasing, and $\theta(x)$ goes to zero as x goes to zero. Therefore Proposition IV.1 and the remark that follows show that

$$\|T^{2k}\|_{1\to\infty} \leqslant m_k, \qquad \forall k \in \mathbb{N}^*,$$

where the sequence m_k is defined by $m_1 = \|T^2\|_{1\to\infty}$ and $\theta(m_{k+1}) = m_k - m_{k+1}$. We have to show that there exists C, $c > 0$ such that $m_k \leqslant Cm(ck)$, $\forall k \in \mathbb{N}^*$.

Let $\alpha > 0$ be such that $(m'(k+1))/(m'(k)) \geqslant \alpha$, $\forall k \in \mathbb{N}^*$. Take k_0 large enough so that $m_{k_0} \leqslant m(1)$ (one checks easily that the sequence m_k converges to zero) and $\alpha/k_0 \leqslant 1$. Set $m'_k = m_{kk_0}$.

One has

$$\begin{aligned} m'_k - m'_{k+1} &= m_{kk_0} - m_{kk_0+1} + \cdots - m_{kk_0+k_0} \\ &= \theta(m_{kk_0+1}) + \cdots + \theta(m_{kk_0+k_0}) \\ &\geqslant k_0\theta(m_{(k+1)k_0}) = k_0\theta(m'_{k+1}). \end{aligned}$$

We have $m'_1 \leqslant m(1)$. Suppose $m'_k \leqslant m(k)$ and $m'_{k+1} > m(k+1)$. Then

$$\theta(m'_{k+1}) \leqslant \frac{1}{k_0}(m'_k - m'_{k+1}) < \frac{1}{k_0}(m(k) - m(k+1)) = -\frac{1}{k_0} m'(k+\eta),$$

where $\eta \in]0, 1[$, by the mean value theorem, therefore

$$\theta(m'_{k+1}) \leqslant -\frac{\alpha}{k_0} m'(k+1) = \frac{\alpha}{k_0}\theta(m(k+1)) \leqslant \theta(m(k+1)).$$

Since θ is non-decreasing, this contradicts $m'_{k+1} > m(k+1)$. Therefore, $\forall k \in \mathbb{N}^*$, $m_{kk_0} \leqslant m(k)$, which clearly gives what we want.

Remark. In the case where the measured space (X, ξ) is discrete, some simplifications occur in the above considerations: in the conclusion of IV.2, one can take $R = Id$. Conversely, starting in IV.1 from

$$\theta(\|f\|_2^2) \leqslant \|f\|_2^2 - \|Tf\|_2^2, \qquad \forall f \in L^1 \cap L^2, \|f\|_1 = 1,$$

one can define m by $\theta(m(k)) = m(k) - m(k+1)$, and the proof of IV.3, with (ii) replaced by the above condition, is even simpler.

In the spirit of II.5, Theorems IV.1 and IV.2 can also be used to control a non-symmetric kernel by a symmetric one. Moreover, the pointwise domination of the second kernel by the first one is enough to ensure the comparison of Dirichlet forms that is needed.

Let $(p_i)_{i=1,2}$ be two bimarkovian kernels on X, i.e., two measurable mappings from $X \times X$ to $\mathbb{R}_+$, such that

$$\int p_i(x, y)\, d\xi(y) = \int p_i(y, x)\, d\xi(y) = 1, \quad \forall x \in X,$$

$$\text{and } \sup_{x, y} p_i(x, y) < +\infty, \; i = 1, 2.$$

Define $p_i^{(k)}$ by $p_i^{(1)} = p_i$ and $p_i^{(k)}(x, y) = \int p_i^{(k-1)}(x, z)\, p_i(z, y)\, d\xi(z)$; we shall say that $p_1 \ll p_2$ if there exists C such that $p_1(x, y) \leqslant C p_2(x, y)$, $\forall (x, y) \in X \times X$ and that p_1 is symmetric if $p_1(x, y) = p_1(y, x)$, $\forall (x, y) \in X \times X$.

One then has the following generalisation of [CS2], Section II:

IV.4. Proposition. *Let $(p_i)_{i=1,2}$ be two bimarkovian kernels on X such that p_1 is symmetric and $p_1 \ll p_2$, and let m be a C^1 function satisfying $(\tilde{D})$. Then*

$$\sup_{x, y} p_1^{(k)}(x, y) = O(m(k)), \qquad k \in \mathbb{N}^*$$

implies

$$\sup_{x, y} p_2^{(k)}(x, y) = O(m(ck)), \qquad k \in \mathbb{N}^*,$$

for some $c > 0$.

Proof. Let T_1 and T_2 be the operators respectively associated to p_1 and p_2, i.e.,

$$T_i f(x) = \int p_i(x, y)\, f(y)\, d\xi(y), \qquad i = 1, 2.$$

It follows from the assumptions on p_1 and p_2 that T_1 is Markovian and self-adjoint on $L^2(X, \xi)$, and that T_2, T_2^* are Markovian, therefore T_2 is contractive on the $L^p(X, \xi)$, $1 \leqslant p < +\infty$. Moreover, T_1 and T_2 are regularising.

By hypothesis $\|T_1^k\|_{1\to\infty} = \sup_{x,y} p_1^{(k)}(x, y) = 0(m(k))$. Since T_1 is symmetric Markovian, Proposition IV.2 applies with $S = T_2$ and gives

$$\theta(c\,\|T_2 f\|_2^2) \leqslant C(\|f\|_2^2 - \|T_1 f\|_2^2), \qquad \forall f \in L^1 \cap L^2,\ \|f\|_1 \leqslant 1,$$

Let us compare $\|f\|_2^2 - \|T_1 f\|_2^2$ with $\|f\|_2^2 - \|T_2 f\|_2^2$ as in [CS2]; the hypothesis $p_1 \ll p_2$ implies that, for α small enough, $p_2 = \alpha p_1 + (1-\alpha)\, p_3$, when p_3 is again a bimarkovian kernel. It follows that

$$\begin{aligned}\|T_2 f\|_2^2 &\leqslant (\alpha\,\|T_1 f\|_2 + (1-\alpha)\|T_3 f\|_2)^2\\ &\leqslant \alpha\,\|T_1 f\|_2^2 + (1-\alpha)\|T_3 f\|_2^2\\ &\leqslant \alpha\,\|T_1 f\|_2^2 + (1-\alpha)\|f\|_2^2,\end{aligned}$$

since T_3, the operator associated to p_3, is a contraction of $L^2(X)$.

Finally $\|f\|_2^2 - \|T_1 f\|_2^2 \leqslant \alpha^{-1}(\|f\|_2^2 - \|T_2 f\|_2^2)$, and

$$\theta(c\,\|T_2 f\|_2^2) \leqslant C(\|f\|_2^2 - \|T_2 f\|_2^2), \qquad \forall f \in L^1 \cap L^2,\ \|f\|_1 \leqslant 1,$$

therefore $\|T_2^k\|_{1\to\infty} = O(m(k))$, after Proposition IV.1.

V. MARKOV CHAINS ON GRAPHS

Let X be an infinite, connected graph. Write $x \sim y$ if $x, y \in X$ are neighbours. Let $n(x) - 1$ be the number of neighbours of $x \in X$. Assume that X is uniformly locally finite, i.e., $\sup_{x \in X} n(x) < +\infty$. We shall take the l^p norms on X with respect to the measure $n(x)\, dx$, that is equivalent to the counting measure. If f be a finitely supported function on X, one can define the length of its gradient by

$$|\nabla_X f|(x) = |\nabla f|(x) = \sum_{y \in X,\, x \sim y} |f(x) - f(y)|.$$

Denote by $|\Omega|$ the cardinal of the set $\Omega \subset X$, and define $\partial\Omega = \{x \in \Omega;\ \exists y \notin \Omega,\ y \sim x\}$.

Let p be a Markov kernel on X, i.e., a function $p: X \times X \to \mathbb{R}_+$ such that $\sum_{y \in X} p(x, y) = 1$, $\forall x \in X$. We shall say that p is admissible if it is reversible with respect to n, i.e., $p(x, y)\, n(x) = p(y, x)\, n(y)$, if there exists $c > 0$ satisfying $p(x, y) \geqslant c$, if $x, y \in X$ are neighbours or identical and if p has bounded range, i.e., there exists r_0 such that $p(x, y) = 0$ as soon as $d(x, y) \geqslant r_0$. Denote by p_k the kth iterated kernel of p : $p_1 = p$ and $p_k = \sum_{z \in X} p_{k-1}(x, z)\, p(z, y)$.

The standard kernel q, defined by $q(x, y) = 1/(n(x))$ if $y \sim x$, or $y = x$, 0 otherwise, is admissible. If p is admissible and T is the operator defined by

$$Tf(x) = \sum_{y \in X} p(x, y) f(y),$$

one checks easily that T is symmetric Markovian on $\ell^2(X)$ and that

$$C^{-1} \| |\nabla f| \|_2^2 \leqslant \|f\|_2^2 - \|Tf\|_2^2 \leqslant C \| |\nabla f| \|_2^2.$$

The following statement is the discrete version of the main theorem in [G1]. It is formulated up to equivalence of functions in the sense of Section II.

V.1. PROPOSITION. *Let X be as above and let p be an admissible kernel on X; then, if m is a C^1 function satisfying $(\tilde{D})$, one has*

$$\sup_{x, y \in X} p_k(x, y) \leqslant m(k)$$

if and only if

$$\|f\|_2 \leqslant \psi(|\Omega|) \| |\nabla f| \|_2,$$

for every Ω finite subset of X and every function f supported in Ω, where ψ and m are related by

$$t = \int_{m(t)}^{+\infty} \psi^2 \left(\frac{1}{x}\right) \frac{dx}{x}$$

or

$$\psi(x) = \frac{1}{\sqrt{-xm'(m^{-1}(1/x))}}.$$

Proof. According to [BCLS], Section 9.B, the inequality

$$\|f\|_2 \leqslant \psi(|\Omega|) \| |\nabla f| \|_2$$

for every Ω finite subset of X and every function f supported in Ω, is equivalent to

$$\|f\|_2 \leqslant \psi\left(\frac{\|f\|_1^2}{\|f\|_2^2}\right) \| |\nabla f| \|_2$$

for every function f finitely supported in X. Since p is admissible, this means that $\theta(\|f\|_2^2) \leqslant C(\|f\|_2^2 - \|Tf\|_2^2)$, $\forall f \in \ell^1(X)$, $\|f\|_1 = 1$, where $\theta(x) = x/(\psi^2(1/x))$. One checks easily that ψ is non-decreasing, therefore $\lim_{x \to 0^+} \theta(x) = 0$. The conclusion then follow from Proposition IV.3 and the subsequent remark.

Remark. The behaviour of $\psi(x)$ and $m(t)$ does not matter for x or t small, and the convergence of $\int_{m(t)}^{+\infty} \psi^2(1/x)\, dx/x$ is not an issue.

V.2. Corollary. *Let X be as above. Suppose that X satisfies the isoperimetric inequality*

$$\frac{|\Omega|}{\psi(|\Omega|)} \leqslant C\, |\partial\Omega|,$$

for all Ω finite subset of X and let p be an admissible kernel on X, where ψ is non-decreasing; then

$$\sup_{x,\, y \in X} p_k(x, y) \leqslant m(k),$$

where m and ψ are related as above.

Proof. The hypothesis implies

$$\|f\|_1 \leqslant \psi(|\Omega|)\ \| |\nabla f| \|_1$$

for every Ω finite subset of X and every function f supported in Ω. This inequality applied to f^2 together with the Hölder inequality implies

$$\|f\|_2 \leqslant C\psi(|\Omega|)\ \| |\nabla f| \|_2$$

for every Ω finite subset of X and every function f supported in Ω. One can therefore apply the direct part of Proposition V.1.

Remark. Corollary V.2 can of course be combined with Proposition IV.4 in order to treat more general kernels.

Endowed with the natural graph distance, X becomes a metric space. Let $B(x, n)$ be the closed ball of center $x \in X$ and radius $n \in \mathbb{N}$, and let $V(x, n) = |B(x, n)|$ be its cardinal. If f is a function on f and $n \in \mathbb{N}^*$, define f_n by $f_n(x) = (1/V(x, n)) \sum_{y \in B(x, n)} f(y)$.

V.3. Proposition. *Let X be as above. Suppose that X satisfies the volume lower bound*

$$V(x, n) \geqslant V(n), \qquad \forall x \in X, \quad \forall n \in \mathbb{N}^*$$

and the inequality

$$\|f - f_n\|_p \leqslant Cn \, \|\nabla f\|_p, \qquad \forall n \in \mathbb{N}^*, \quad \forall f \in c_0(X),$$

for some $p \in [1, 2]$. *Then*

$$\sup_{x,\, y \in X} p_k(x, y) \leqslant m(k), \qquad k \in \mathbb{N}^*$$

for any admissible kernel p *on* X, *where* ψ *is the inverse function of* V *and* m *is related to* ψ *as above.*

Proof. According to [CS3] (see also [C6]), the above volume lower bound and pseudo-Poincaré inequality yield the hypothesis of Corollary V.2.

Remark. In particular, if the growth function of a finitely generated group G is V, one has

$$\mu^{(k)}(e) \leqslant m(k),$$

where e is the neutral element in G and $\mu^{(k)}$ is the kth convolution power of μ, μ being for example a symmetric probability on G whose support is finite and generating. This result was already known in all the known cases for V, namely $V(n) \approx n^D$ and $V(n) \geqslant ce^{n^\alpha}$ (see [VSC] and [V3]). The above statement can also be used in the setting of almost-transitive Markov chains, in the spirit of [S].

Let us end this section with a generalisation of [CS3], thm. 7. The following lemma is due to Saloff-Coste [S]. Notice that it does not introduce any doubling volume condition.

V.4. Lemma. *Let* V_1, V_2 *be such that*

$$C^{-1} V_1(n) \leqslant V(x, n) \leqslant C V_2(n), \qquad \forall x \in X, \quad \forall n \in \mathbb{N}^*.$$

Then

$$\|f - f_n\|_2 \leqslant C \left(\frac{n}{V_1(n)} \right)^{1/2} V_2(n) \, \|\nabla f\|_2, \qquad \forall n \in \mathbb{N}^*, \quad \forall f \in c_0(X).$$

Proof. Let $x \in X$ and $n \in \mathbb{N}^*$; for any $y \in B(x, n)$, choose a minimising path $\gamma_{x,\, y}$ joining x to y. Set $\Gamma_{x,\, n} = \{\gamma_{x,\, y} \mid y \in B(x, n)\}$. One has

$$\begin{aligned} \|f - f_n\|_2^2 &= \sum_x \left| f(x) - \frac{1}{V(x, n)} \sum_{y \in B(x, n)} f(y) \right|^2 \\ &\leqslant \sum_x \frac{1}{V(x, n)} \sum_{y \in B(x, n)} |f(x) - f(y)|^2. \end{aligned}$$

Now, by Cauchy-Schwarz,

$$|f(x)-f(y)|^2 \leqslant |\gamma_{x,y}| \sum_{e \in \gamma_{x,y}} |f(e_+)-f(e_-)|^2,$$

for $y \in B(x, n)$, e_+ and e_- being the extremities of the edge e. It follows that

$$\sum_{y \in B(x,n)} |f(x)-f(y)|^2 \leqslant \sum_{e \in B(x,n)} \sum_{\{\gamma \in \Gamma_{x,n} \mid e \in \gamma\}} |\gamma| \, |f(e_+)-f(e_-)|^2.$$

Here $e \in \gamma$ means that e_+ or e_- belong to $B(x, n)$. Observe now that

$$\max_{e \in B(x,2n)} \sum_{\{\gamma \in \Gamma_{x,n} \mid e \in \gamma\}} |\gamma| \leqslant n V_2(n).$$

Therefore

$$\|f-f_n\|_2^2 \leqslant n \frac{V_2(n)}{V_1(n)} \sum_x \sum_{e \in B(x,n)} |f(e_+)-f(e_-)|^2.$$

Now

$$\sum_x \sum_{e \in B(x,n)} |f(e_+)-f(e_-)|^2 \leqslant C(n) \, \|\nabla f\|_2^2,$$

where $C(n)$ is the maximum number of overlappings between the balls $B(x, n)$, $x \in X$. But $V_2(n)$ is clearly an upper bound for $C(n)$. The lemma is proved.

Remarks. — One can always choose $V_1(n) = \inf_{x \in X} V(x, n) \geqslant n$ and $V_2(n) = \sup_{x \in X} (x, n) \leqslant N^n$, where $N = \sup_{x \in X} n(x) < +\infty$.

— When the volume growth of X is uniform in the sense that there exists V such that

$$C^{-1} V(n) \leqslant V(x, n) \leqslant C V(n), \qquad \forall x \in X, \quad \forall n \in \mathbb{N}^*,$$

one gets

$$\|f-f_n\|_2 \leqslant C[nV(n)]^{1/2} \, \|\nabla f\|_2, \qquad \forall n \in \mathbb{N}^*, \quad \forall f \in c_0(X).$$

V.5. Proposition. *Let X be any infinite connected graph, let V_1, V_2 be such that*

$$C^{-1} V_1(n) \leqslant V(x, n) \leqslant C V_2(n), \qquad \forall x \in X, \quad \forall n \in \mathbb{N}^*,$$

and p an admissible kernel on X. Then

$$\sup_{x,\, y \in X} p_k(x, y) \leqslant m(k),$$

where m is given by

$$t = \int_{m(t)}^{+\infty} (V_2 \circ \psi(1/x))^2\, \psi(1/x)\, dx$$

and ψ is the inverse function of V_1.

Proof. Write

$$\|f\|_2^2 = (f, f - f_n) + (f, f_n) \leqslant \|f\|_2\, \|f - f_n\|_2 + \|f_n\|_\infty\, \|f\|_1$$

$$\leqslant C\, \|f\|_2 \left(\frac{n}{V_1(n)}\right)^{1/2} V_2(n)\, \|\nabla f\|_2 + \frac{C'}{V_1(n)}\, \|f\|_1^2,$$

and choose $n = \psi(2C' \|f\|_1^2 / \|f\|_2^2)$. One gets the Nash inequality

$$\theta(\|f\|_2^2) \leqslant \|\nabla f\|_2^2, \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 = 1,$$

with $\theta(x) = 1/(V_2 \circ \psi(1/x))^2\, \psi(1/x))$. The result follows from Proposition V.1.

Examples. — When there exists V such that

$$C^{-1} V(n) \leqslant V(x, n) \leqslant CV(n), \qquad \forall x \in X, \forall n \in \mathbb{N}^*,$$

one gets

$$t = \int_{m(t)}^{+\infty} \psi\left(\frac{1}{x}\right) \frac{dx}{x^2},$$

ψ being the inverse function of V. Compare with V.1., where ψ may be thought of as the inverse of a volume growth function.

In the case where $V(n) = e^{cn}$, one finds

$$\sup_{x,\, y \in X} p_k(x, y) = O\left(\frac{\log k}{k}\right).$$

— Suppose that

$$C^{-1} n^a \leqslant V(x, n) \leqslant Cn^b, \qquad \forall x \in X, \forall n \in \mathbb{N}^*,$$

$1 \leqslant a \leqslant b$. One finds

$$\sup_{x,\, y \in X} p_k(x, y) = O(k^{-a/(2b-a+1)}).$$

The case $a = b$ was treated in [CS3], Thm. 7.

Remark. From V.2 on, we do not really use the new idea that is the core of this paper, since we invoke the direct statement IV.1 and not the converse IV.2. However, the fact that V.1 is an equivalence tells us how far from (or close to) the conclusion are our assumptions on the isoperimetry or the volume.

VI. DISCRETISATION

Let M be a complete and connected Riemannian manifold, endowed with its canonical measure dx. Let $B(x, r)$ be the geodesic ball of center $x \in M$ and radius r, and $V(x, r)$ its volume. We shall say that M satisfies $(DV)_0$ if $\forall r_0 > 0$, there exists $C(r_0)$ such that,

$$\forall x \in M, \quad \forall r \leqslant r_0, \qquad V(x, 2r) \leqslant C(r_0)\, V(x, r).$$

If $\psi \in C_0^\infty(M)$ and $r > 0$ define ψ_r by

$$\psi_r(x) = \frac{1}{V(x, r)} \int_{B(x, r)} \psi(y)\, dy, \qquad x \in M.$$

We shall say that M satisfies $(P)_0$ if, $\forall r_0 > 0$, there exists $C'(r_0)$ such that, $\forall \psi \in C_0^\infty(M)$, $x \in M$, and $r \leqslant r_0$,

$$\int_{B(x, r)} |\psi(y) - \psi_r(x)|^2\, dy \leqslant C'(r_0)\, r^2 \int_{B(x, 2r)} |\nabla \psi(y)|^2\, dy.$$

This terminology was introduced in [CS4]. Note that manifolds M with Ricci curvature bounded from below satisfy $(DV)_0$ and $(P)_0$. The class of manifolds satisfying $(DV)_0$ and $(P)_0$ is close to the class of locally Harnack manifolds considered in [G2]. Suppose in addition that M satisfies $\inf_{x \in M} V(x, r) > 0$, $\forall r > 0$, which is the case if M has positive injectivity radius. Then $\sup_{x,\, y \in M} p_t(x, y) < +\infty$, $t > 0$, where p_t is the heat kernel on M. In other terms, the heat semigroup is ultracontractive.

Fix $\varepsilon > 0$. Let X be a maximal ε-separated subset of M: $d(x, y) \geqslant \varepsilon$, $\forall x, y \in X$, $x \neq y$ and $d(x, X) < \varepsilon$, $\forall x \in M$. We shall say that X is a discretisation of M. Let us decide that two points in X are neighbours if their Riemannian distance in M is smaller than 2ε. This way X becomes a uniformly locally finite connected graph. The idea of this construction comes from [K] in the case of manifolds with Ricci curvature bounded from below and positive injectivity radius, and it adapts easily to our more general setting (see [CS4]).

In the sequel, we shall associate a function $\tilde{\psi}$ on X with a function ψ on M by setting, for $x \in X$,

$$\tilde{\psi}(x) = \psi_\varepsilon(x) = \frac{1}{V(x, \varepsilon)} \int_{B(x, \varepsilon)} \psi(y)\, dy.$$

We shall also associate a function $\hat{f}$ on M with a function f on X by setting

$$\hat{f}(y) = \sum_{x \in X} f(x)\, \phi_x(y), \qquad y \in M,$$

where $(\phi_x)_{x \in X}$ is a suitable partition of unity on M. We shall also consider the operator S from $C_0^\infty(M)$ to itself defined by $S\psi = (\tilde{\psi})^\wedge$. All these notions have been introduced in [C3] and studied in detail in [CS4].

The following statement generalises the main result of [C3]. It is formulated up to equivalence of functions.

VI.1. Theorem. *Let M be a complete and connected Riemannian manifold satisfying $(DV)_0$, $(P)_0$ and $\inf_{x \in M} V(x, 1) > 0$. Let X be a discretisation of M, and m a C^1 function satisfying condition $(\tilde{D})$. Then the two following conditions are equivalent*:

(i) $\sup_{x, y \in M} p_t(x, y) \leqslant m(t)$, $\forall t \geqslant 1$.

(ii) *The standard Markov chain on X, with kernel q, satisfies* $\sup_{x, y \in X} q_k(x, y) \leqslant m(k)$, $\forall k \in \mathbb{N}^*$.

Proof. According to III.3, (i) is equivalent to

$$\theta(\|\psi\|_2^2) \leqslant C(-\Delta\psi, \psi) = C \,\| |\nabla\psi| \|_2^2, \qquad \forall \psi \in C_0^\infty(M),\ \|\psi\|_1 \leqslant 1,\ \|\psi\|_2 \leqslant a.$$

In particular,

$$\theta(\|\hat{f}\|_2^2) \leqslant C \,\| |\nabla\hat{f}| \|_2^2, \qquad \forall f \in C_0(X),\ \|\hat{f}\|_1 \leqslant 1,\ \|\hat{f}\|_2 \leqslant a.$$

Now, if $|\nabla_X|$ is the discrete length of the gradient defined in Section V,

$$\| |\nabla\hat{f}| \|_2^2 \leqslant C \,\| |\nabla_X f| \|_2^2,$$

([CS4], Lemme 6.4) hence

$$\| |\nabla\hat{f}| \|_2^2 \leqslant C'(\|f\|_2^2 - \|Tf\|_2^2).$$

Now $\|f\|_1 \leqslant b$ implies $\|f\|_2 \leqslant b$ hence, for b small enough, $\|\hat{f}\|_2 \leqslant a$ ([CS4], Lemme 6.2). Therefore

$$\theta(c\,\|f\|_2^2) \leqslant C(\|f\|_2^2 - \|Tf\|_2^2), \qquad \forall f \in c_0(X),\ \|f\|_1 < 1.$$

According to IV.1, (ii) follows.

Let us now assume (ii). Since X is discrete, $R = Id$ is regularising and, according to IV.2,

$$\theta(c\,\|f\|_2^2) \leqslant C(\|f\|_2^2 - \|Tf\|_2^2), \qquad \forall f \in c_0(X),\ \|f\|_1 \leqslant 1.$$

In particular,

$$\theta(c\,\|\tilde{\psi}\|_2^2) \leqslant C(\|\tilde{\psi}\|_2^2 - \|T\tilde{\psi}\|_2^2) \leqslant \|\,|\nabla_X \tilde{\psi}|\,\|_2^2, \qquad \forall \psi \in C_0^\infty(M),\ \|\tilde{\psi}\|_1 \leqslant 1.$$

It follows, using [CS4], Lemme 6.3 and Lemme 6.4, that

$$\theta(c\,\|\tilde{\psi}\|_2^2) \leqslant C\,\|\ |\nabla\psi|\ \|_2^2, \qquad \forall \psi \in C_0^\infty(M),\ \psi \geqslant 0,\ \|\psi\|_2 \leqslant c.$$

But

$$\begin{aligned}\|\psi\|_2^2 &\leqslant 2(\|S\psi\|_2^2 + \|\psi - S\psi\|_2^2)\\ &\leqslant C\,\|\tilde{\psi}\|_2^2 + C'\,\|\,|\nabla\psi|\,\|_2^2.\end{aligned}$$

Here we have used [CS4], Lemme 6.3 and Lemme 5.2. It follows that

$$\|\psi\|_2^2 \leqslant C\theta^{-1}(\|\,|\nabla\psi|\,\|_2^2) + C'\,\|\,|\nabla\psi|\,\|_2^2.$$

We have seen in the proof of IV.2 that if $x \leqslant a^2$, $\theta(x) \leqslant Cx$, hence $\theta(\theta^{-1}(x) + x) \leqslant C'x$. Therefore, if $\|\psi\|_2^2 \leqslant a^2$,

$$\theta(c\,\|\psi\|_2^2) \leqslant C\,\|\,|\nabla\psi|\,\|_2^2, \qquad \forall \psi \in C_0^\infty(M),\ \psi \geqslant 0,\ \|\psi\|_1 \leqslant 1.$$

According to III.1, this implies (i).

Now we can deduce from Theorem VI.1 and Proposition V.5 the following generalisation of [CS3], Théorème 8.

VI.2. Corollary. *Let M be a complete and connected Riemannian manifold satisfying $(DV)_0$, $(P)_0$, such that*

$$C^{-1}V_1(r) \leqslant V(x, r) \leqslant CV_2(r), \qquad \forall x \in M, \quad \forall r \geqslant 1.$$

Then

$$\sup_{x,\, y \in X} p_t(x, y) = O(m(ct)), \qquad t \to +\infty,$$

where m is given by

$$t=\int_{m(t)}^{+\infty}(V_2\circ\psi(1/x))^2\,\psi\left(\frac{1}{x}\right)dx$$

and ψ is the inverse function of V_1.

EXAMPLES. If $V_1(r)=r^a$ and $V_2(r)=r^b$, one finds

$$\sup_{x,\,y\in X}p_t(x,y)=O(t^{-[a/(2b-a+1)]}),\qquad t\to+\infty.$$

If M has uniform exponential growth, i.e., $V_1(r)=V_2(r)=e^{cr}$, one finds

$$\sup_{x,\,y\in X}p_t(x,y)=O\left(\frac{\log t}{t}\right),\qquad t\to+\infty.$$

From Theorem VI.1 and the Remark following Proposition V.3, one deduces

VI.3. PROPOSITION. *Let M_1 be a compact Riemannian manifold and M a Galois covering of M_1. Let p_t be the heat kernel on M, G the group of the covering, and V the volume growth function of G. Then*

$$\sup_{x,\,y\in M}p_t(x,y)=O(m(ct)),\qquad t\to+\infty,$$

where m is defined by

$$t=\int_{m(t)}^{+\infty}\psi^2\left(\frac{1}{x}\right)\frac{dx}{x},$$

ψ being the inverse function of V.

This statement is contained in [V2] in the case of polynomial growth. In the general case, it also follows from [CS3] and [G1].

Finally, from Theorem VI.1, Proposition V.1 and the fact that a Nash type inequality is preserved between quasi-isometric graphs, one deduces

VI.4. THEOREM. *Let M_1, M_2 be two roughly isometric Riemannian manifolds satisfying $(P)_0$, $(DV)_0$ and $\inf_{x\in M}V_i(x,1)>0$, $i=1,2$. Let p_t^i be the heat kernels on M_i, $i=1,2$. Suppose that*

$$\sup_{x,\,y}p_t^1(x,y)\leqslant m(t),\qquad t\geqslant 1$$

where m is C^1 and satisfies $(\tilde{D})$. Then there exists C, c such that

$$\sup_{x,\,y} p_t^2(x, y) \leqslant Cm(ct), \qquad t \geqslant 1.$$

The case of quasi-isometric manifolds was already treated in [G1].

Theorem VI.4 can be generalised to roughly isometric subelliptic operators in the spirit of [CS4], Section 9: in [CS4], Proposition 9.4, 2, one can replace $t^{-\nu/2}$ by any $m(t)$ satisfying (D).

APPENDIX: ONE-PARAMETER SOBOLEV INEQUALITIES

We develop here briefly an idea that comes from [V3]. Let T_t be a symmetric submarkovian semigroup with infinitesimal generator $-A$. Suppose there exists φ such that

$$\|f\|_{2+\varepsilon} \leqslant \varphi(\varepsilon)(Af, f)^{1/2}, \qquad \forall f \in \mathscr{D}(A), \quad \forall \varepsilon > 0.$$

Via Hölder, this gives

$$\|f\|_2 \leqslant (\varphi(\varepsilon)(Af, f)^{1/2})^{(2+\varepsilon)/2(1+\varepsilon)}, \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1.$$

Therefore

$$\theta(\|f\|_2^2) \leqslant (Af, f), \qquad \forall f \in \mathscr{D}(A),\ \|f\|_1 \leqslant 1,$$

with $\theta(x) = \sup_{\varepsilon > 0} (x^{2(1+\varepsilon)/(2+\varepsilon)}/(\varphi^2(\varepsilon))$. From II.1, one concludes that

$$\|T_t\|_{1 \to \infty} \leqslant m(t), \qquad \forall t > 0,$$

where $t = \int_{m(t)}^{+\infty} dx/\theta(x)$.

Conversely, suppose T_t is ultracontractive, and

$$\|T_t\|_{1 \to \infty} \leqslant m(t), \qquad \forall t > 0.$$

Then

$$\|T_t\|_{1 \to \infty} \leqslant \varphi(\varepsilon)\, t^{-1-2/\varepsilon}, \qquad \forall t > 0,$$

where $\varphi(\varepsilon) = \sup_{t>0} (t^{1+(2/\varepsilon)} m(t))$ and from [V1], it follows that

$$\|f\|_{2+\varepsilon} \leqslant C\varphi(\varepsilon)(Af, f)^{1/2}, \qquad \forall f \in \mathscr{D}(A).$$

The class of functions m for which one loses nothing on the way from m to φ and back to m is not clear, but certainly includes interesting examples (the case $m(t) = e^{-t^\alpha}$, $0 < \alpha < 1$, is treated in [V3]).

ACKNOWLEDGMENTS

The author thanks Françoise Demengel, Alexander Grigor'yan, Derek Robinson, and Laurent Saloff-Coste for useful conversations.

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